Math 72 4.5-1st Solving Linear Systems in 3 variables

Objectives

1) Solve a system of 3 equations in 3 variables using elimination and/or substitution

2) Recognize a system which is
   - inconsistent (no solution)
   - consistent dependent (many solutions)
   - consistent independent (one solution)

3) Write solutions in appropriate format
   - ordered triple with numbers
   - ordered triple with a parameter (variable)
   - no solution

Solve and classify.

① $\begin{cases} 2x + 4y = 3 & (A) \\ 4x - 4z = -1 & (B) \\ y - 4z = -3 & (C) \end{cases}$

Notice the gaps: It helps to re-write and line up like terms.

$$\begin{cases} 2x + 4y + \underline{\underline{0z}} = 3 & (A) \\ 4x + \underline{\underline{0y}} - 4z = -1 & (B) \\ \underline{\underline{0x}} + y - 4z = -3 & (C) \end{cases}$$

GOALS: Use all 3 equations eventually.
Almost always use 2 equations at a time.

<u>OPTION 1</u>:

- Use (A) & (B) and eliminate x to get (D) { mult (A) by -2 }
- Use (D) and (C) as a 2x2 system

<u>OPTION 2</u>:

- Use (A) & (C) and eliminate y to get (E) { mult (C) by -4 }
- Use (B) & (E) as a 2x2 system.

* <u>OPTION 3</u>:

- Use (B) & (C) and eliminate z to get (F) { mult by (B) by -1 or mult (C) by -1 }
- Use (A) & (F) as a 2x2 system.

* <u>OPTION 4</u>:

- Solve (C) for y
- Subst result into (A) to remove y (distribute and combine)
- Use modified (A) and (B) as a 2x2 system.

<u>OPTIONS 5-9</u>:

- Solve some other equation for some other variable, get fractions.
- Subst into another equation.
- Use that result with 3rd equation as a 2x2 system.

OPTION 3:

mult Ⓑ by -1:

$$\begin{array}{rrrcrl} -4x & + 0y & + \cancel{4z} & = & 1 & \text{(B)} \times (-1) \\ 0x & + y & - \cancel{4z} & = & -3 & \text{(C)} \\ \hline \end{array}$$

$$\begin{cases} -4x + y = -2 & \text{(F)} \\ 2x + 4y = 3 & \text{(A)} \end{cases}$$

more options:

★ • elim x by mult Ⓐ by 2

OR • elim y by mult Ⓕ by -4

OR • solve Ⓕ for y and subst into A

OR • solve either equation for any other variable (get fractions) and subst

I choose to eliminate x by multiplying Ⓐ by 2:

$$\begin{array}{rrcrl} -4\cancel{x} & + y & = & -2 & \text{(F)} \\ \cancel{4}x & + 8y & = & 6 & \text{(A)} \times 2 \\ \hline & 9y & = & 4 & \\ & y & = & \frac{4}{9} & \end{array}$$

more options:

★ • subst y into Ⓕ and solve for x.

• subst y into Ⓐ and solve for x.

★ • subst y into Ⓒ and solve for z.

subst $y=\frac{4}{9}$ into Ⓕ

$$-4x + \frac{4}{9} = -2$$

$$-4x = -2 - \frac{4}{9} = \frac{-22}{9}$$

$$x = \frac{-22}{9} \cdot \frac{-1}{4} = \frac{11}{18}$$

subst $y=\frac{4}{9}$ into Ⓒ

$$\frac{4}{9} - 4z = -3$$

$$-4z = -3 - \frac{4}{9} = \frac{-31}{9}$$

$$z = \frac{-31}{9} \cdot \frac{-1}{4} = \frac{31}{36}$$

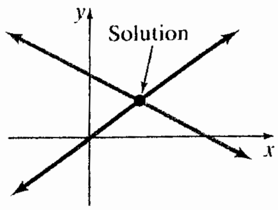
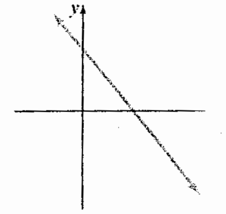
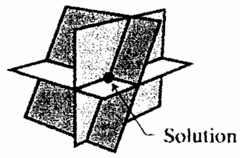
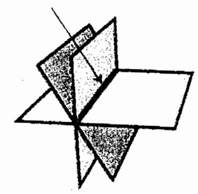
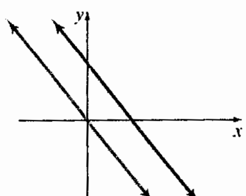
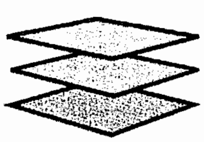
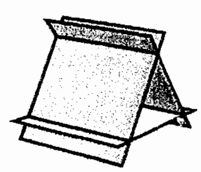
write answer as ordered triple (x, y, z)

$$\boxed{\left(\frac{11}{18}, \frac{4}{9}, \frac{31}{36} \right)}$$

classify

$\boxed{\text{consistent independent}}$

Math 70 Interpreting Results from Solving Linear Systems Using Matrices and RREF on GC

	INDEPENDENT $0=0$ does not appear in RREF	**DEPENDENT** $0=0$ appears in RREF
CONSISTENT System has solution(s)	2 variables $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}$ Consistent Independent Solution (a,b) 	2 variables $\begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \end{bmatrix}$ Consistent Dependent Solutions $\{(x,y): x+ay=b\}$ 
	3 variables $\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$ Consistent Independent Solution (a,b,c) 	3 variables Solutions (line) $\begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 0 & 0 \end{bmatrix}$  Consistent Dependent Solutions $\{(x,y,z) \mid x=b-az, y=d-cz, z \in R\}$ or $(b-az, d-cz, z)$ where $z \in R$
		3 variables Solutions (plane) $\begin{bmatrix} 1 & a & b & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$  Consistent Dependent Solutions $\{(x,y,z) \mid x+ay+bz=c\}$
INCONSISTENT Systems has no solution. 0=1 appears in RREF	2 variables $\begin{bmatrix} 1 & 0 & a \\ 0 & 0 & 1 \end{bmatrix}$ Inconsistent (Independent) No solution { } 	2 variables Inconsistent Dependent is not possible
	3 variables $\begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$  Inconsistent Independent No Solution { }	3 variables $\begin{bmatrix} 1 & a & b & c \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$  Inconsistent Dependent No Solution { }

(2)
$$\begin{cases} 3x - 3y + z = -1 & (A) \\ 3x - y - z = 3 & (B) \\ -6x + 2y + 2z = -6 & (C) \end{cases}$$

option 1: eliminate x twice, using all 3 equations
$\begin{cases} A \ \& \ B \\ B \ \& \ C \end{cases}$ or $\begin{cases} A \ \& \ B \\ A \ \& \ C \end{cases}$ or $\begin{cases} A \ \& \ C \\ B \ \& \ C \end{cases}$

option 2: eliminate y twice
same 3 combinations

★ option 3: eliminate z twice
same 3 combinations

Then solve the resulting 2x2 system

option 4: Solve (A) for z, subst result into (B) & (C)
Solve (B) & (C) as a 2x2 system.

option 5: solve (B) for y, subst result into (A) & (C)
solve (A) & (C) as a 2x2 system.

option 6: solve (B) for z, subst result into (A) & (C)
solve (A) & (C) as a 2x2 system.

options 7-12: solve any equation for any variable, (get fractions)
subst result into other two equations
Solve results as a 2x2 system.

I choose OPTION 3: Eliminate z twice, (A) & (B) and (B) & (C)

$$\begin{array}{rl} 3x - 3y + z = -1 & (A) \\ 3x - y - z = 3 & (B) \\ \hline 6x - 4y \quad\quad = 2 & (D) \end{array}$$

$$\begin{array}{rl} \cancel{6x} - \cancel{2y} - \cancel{2z} = \cancel{6} & (B) \times 2 \\ \cancel{-6x} + \cancel{2y} + \cancel{2z} = \cancel{-6} & (C) \\ \hline 0 = 0 & (E) \end{array}$$

The variables all disappeared!

ASK: Is the result <u>true</u>?

ANSWER: Yes, $0=0$ is true.

This system has many solutions.

Equations B and C are actually the same equation.
Equation A is different from B and C.

Now that we know there are infinitely many solutions, we'll start again and eliminate x using A and B. (or A & C)

$$\begin{array}{ll} \text{A} & 3x - 3y + z = -1 \\ \text{B} & 3x - y - z = 3 \end{array}$$

mult B by -1 to elim x.

$$\begin{array}{ll} \text{A} & \cancel{3x} - 3y + z = -1 \\ \text{B} \times -1 & \cancel{-3x} + y + z = -3 \\ \hline & -2y + 2z = -4 \end{array}$$

Solve for y

$$-2y = -2z - 4$$
$$y = z + 2$$

This gives y <u>in terms of</u> z

We will use z as the <u>parameter</u>.

We start again and eliminate y using A & B (or A & C)

$$\begin{array}{ll} \text{A} & 3x - \cancel{3y} + z = -1 \\ \text{B} \times (-3) & -9x + \cancel{3y} + 3z = -9 \\ \hline & -6x + 4z = -10 \end{array}$$

solve for x

$$-6x = -4z - 10$$
$$x = \tfrac{2}{3}z + \tfrac{5}{3}$$

This gives x <u>in terms of</u> z

Solution $\boxed{\left(\tfrac{2}{3}z + \tfrac{5}{3},\ z+2,\ z\right)}$

We always use z as the parameter.

Why is $\left(\frac{2}{3}z + \frac{5}{3},\ z+2,\ z\right)$ the solution?

Remember, there are infinitely many solutions, but not every ordered triple works. (0, 0, 0) doesn't work in <u>any</u> of the equations, for example.

To get an ordered triple that works, we need the right relationships between the values of x, y and z.

choose your favorite #, Subst. for z above:

★ I choose $6 = z$

$$\left.\begin{aligned} x &= \tfrac{2}{3}(6) + \tfrac{5}{3} = \tfrac{17}{3} \\ y &= (6) + 2 = 8 \\ z &= 6 \end{aligned}\right\}$$

ordered triple $\left(\frac{17}{3}, 8, 6\right)$

Check this ordered triple in each equation

Ⓐ $3\left(\frac{17}{3}\right) - 3(8) + 6 = 17 - 24 + 6 = -1$ ☺

Ⓑ $3\left(\frac{17}{3}\right) - (8) - (6) = 17 - 8 - 6 = 3$ ☺

Ⓒ $-6\left(\frac{17}{3}\right) + 2(8) + 2(6) = -34 + 16 + 12 = -6$ ☺

The parametric ordered triple

$$\boxed{\left(\frac{2}{3}z + \frac{5}{3},\ z+2,\ z\right)}$$

gives a recipe for finding a solution to all 3 equations.

classify: $\boxed{\text{consistent dependent}}$

Solve and classify

3 $\begin{cases} -6x + 12y + 3z = -6 & \text{A} \\ 2x - 4y - z = 2 & \text{B} \\ -x + 2y + \frac{z}{2} = -1 & \text{C} \end{cases}$

Many options, but must
- use all 3 equations
- use elimination or substitution to get a 2x2 system

I choose: eliminate x.

$$\begin{array}{rl} -6\cancel{x} + 12\cancel{y} + 3\cancel{z} = -\cancel{6} & \text{A} \\ \cancel{6}x - \cancel{12}y - \cancel{3}z = \cancel{6} & \text{B} \times 3 \\ \hline 0 = 0 & \end{array}$$

The system is dependent. A & B are the same equation

Eliminate x using C

$$\begin{array}{rl} 2\cancel{x} - 4\cancel{y} - \cancel{z} = \cancel{2} & \text{B} \\ -\cancel{2}x + \cancel{4}y + \cancel{z} = -\cancel{2} & \text{C} \\ \hline 0 = 0 & \end{array}$$

B & C are <u>also</u> the same equation.

We cannot solve for y in terms of z or x in terms of z.

No ordered triple is possible. We must use set notation.

$$\boxed{\{(x,y,z) : \underbrace{2x - 4y - z = 2}\}}$$

The set of ↑ ordered triples ↑ So that ↑ (The simplest) equation is true. ↑

⇑ Could use any equation.

$$\boxed{\begin{array}{l}\text{Consistent} \\ \text{Dependent}\end{array}}$$

(4) $\begin{cases} 6x - 3y + 12z = 4 & (A) \\ -6x + 4y - 2z = 7 & (B) \\ -2x + y - 4z = 3 & (C) \end{cases}$

I choose to eliminate x.

$$\begin{array}{rl} \cancel{6x} - 3y + 12z = 4 & (A) \\ \cancel{-6x} + 4y - 2z = 7 & (B) \\ \hline y + 10z = 11 & (D) \end{array}$$

Eliminate x again, must use (C)

$\begin{cases} -2x + y - 4z = 3 & (C) \\ 6x - 3y + 12z = 4 & (A) \end{cases}$

$$\begin{array}{rl} \cancel{-6x} + \cancel{3y} - \cancel{12z} = 9 & (C) \times 3 \\ \cancel{6x} - \cancel{3y} + \cancel{12z} = 4 & (A) \\ \hline 0 = 13 & \end{array}$$

This is false.

The system is inconsistent

$\boxed{\text{No solution}}$

$\boxed{\text{Inconsistent}}$

(5) $\begin{cases} -x + 2y - 3z = 4 & \text{(A)} \\ 2x - 4y + 6z = 8 & \text{(B)} \\ x - 2y + 3z = 5 & \text{(C)} \end{cases}$

I choose to eliminate x twice.

$$\begin{array}{rl} -\cancel{2x} + \cancel{4y} - \cancel{6z} = 8 & \text{(A)}\times 2 \\ \cancel{2x} - \cancel{4y} + \cancel{6z} = 8 & \\ \hline 0 = 16 & \end{array}$$

System is $\boxed{\text{inconsistent}}$

$\boxed{\text{no solution}}$

(6) $\begin{cases} x - 5y = 0 & \text{(A)} \\ x - z = 0 & \text{(B)} \\ -x + 5z = 0 & \text{(C)} \end{cases}$

Rewrite lining up like terms:

$$\begin{cases} x - 5y \qquad = 0 & \text{(A)} \\ x \qquad - z = 0 & \text{(B)} \\ -x \qquad + 5z = 0 & \text{(C)} \end{cases}$$

If we use (B) and (C) to eliminate x, we can solve for z!

$$\begin{array}{rl} \cancel{x} - z = 0 \\ \cancel{-x} + 5z = 0 \\ \hline 4z = 0 \\ z = 0 \end{array}$$

$x - z = 0$ (B)

$x - 0 = 0$

$x = 0$

$0 - 5y = 0$ (A)

$-5y = 0$

$y = 0$

Solution $\boxed{(0,0,0)}$ $\boxed{\text{consistent independent}}$

7 $\begin{cases} x - 4y - 5z = 35 & \text{A} \\ x - 3y = 0 & \text{B} \\ -y + z = -55 & \text{C} \end{cases}$

I choose to eliminate x.

$$\begin{array}{rl} \cancel{x} - 4y - 5z = 35 & \text{A} \\ \cancel{-x} + 3y \quad\quad = 0 & \text{B} \times (-1) \\ \hline -y - 5z = 35 & \text{D} \end{array}$$

Solve 2x2 solution from C and D

$\begin{cases} -y + z = -55 & \text{C} \\ -y - 5z = 35 & \text{D} \end{cases}$

Eliminate y.

$$\begin{array}{rl} \cancel{y} - z = 55 & \text{C} \times -1 \\ \cancel{-y} - 5z = 35 & \text{D} \\ \hline -6z = 90 & \end{array}$$

$z = \frac{90}{-6} = -15$

Subst $z = -15$ into C to solve for y.

$-y + (-15) = -55$

$-y = -40$

$y = 40$

Subst $y = 40$ (and $z = -15$) into either A or B to find x.

I choose B

$x - 3(40) = 0$

$x - 120 = 0$

$x = 120.$

Solution $\boxed{(120, 40, -15)}$

Classify $\boxed{\text{Consistent independent}}$